### Governing Equations

The governing dynamical equations may be written in the form[1]

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} = \rho a_i \quad (1)$$

where $u_i$ and $a_i$ are, respectively, the displacement and acceleration of a typical point of the elastic medium The notation employed is the conventional Cartesian tensor notation, with the comma denoting partial differentiation with respect to the space variables indicated by the subsequent subscripts The acceleration must be computed in a Newtonian reference frame and may be expressed as

$$a_i = (\partial^2 u_i/\partial t^2) + e_{ijk}e_{klm}\Omega_j\Omega_l x_m + 2e_{ijk}\Omega_j(\partial u_k/\partial t) \quad (2)$$

where the Cartesian coordinate system $x_i$ is considered to be rotating with uniform angular velocity $\Omega_i$ The partial time derivatives are computed with respect to the rotating system, and $e_{ijk}$ is the Cartesian alternating tensor The development of Eq (2) follows from rewriting in tensor form the general expression for acceleration found in Kane[2] and using the assumptions of uniform rotation and small displacement

Substituting Eq (2) into Eq (1) leads to the equations

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} - \rho(\partial^2 u_i/\partial t^2) - 2\rho e_{ijk}\Omega_j(\partial u_k/\partial t) = \rho e_{ijk}e_{klm}\Omega_j\Omega_l x_m \quad (3)$$

Hence, the homogeneous equations of motion are

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} - \rho(\partial^2 u_i/\partial t^2) - 2\rho e_{ijk}\Omega_j(\partial u_k/\partial t) = 0 \quad (4)$$

### Wave Propagation

The technique used for the investigation of wave propagation is that of Synge[3] in his work on the motion of viscous fluids conducting heat Following this technique, solutions of Eq (4) are taken in the form

$$u_i = A_i \exp(a_k x_k + bt) \quad (5)$$

where the $A_i$ are constants, and $a_i$ and $b$ are complex constants that may be expressed as

$$a_k = a_k' + ia_k'' \qquad b = b' + ib'' \quad (6)$$

where the primed quantities are real constants The equations

$$a_k'x_k + b't = \text{const} \qquad a_k''x_k + b''t = \text{const} \quad (7)$$

represent, respectively, "amplitude" waves and "phase" waves or vibrating motion The squares of the respective propagation velocities are given by

$$(V')^2 = (b')^2/(a_k'a_k')^2 \qquad (V'')^2 = (b'')^2/(a_k''a_k'')^2 \quad (8)$$

Substituting Eq (5) into Eq (4) leads to

$$(\lambda + \mu)A_k a_k a_i + (\mu a^2 - \rho b^2)A_i - 2\rho b e_{ijk}\Omega_j A_k = 0 \quad (9)$$

where

$$a^2 = a_k a_k \quad (10)$$

Consider now waves propagated in planes normal to the axis of rotation If this axis is taken to be $x_3, a_3$ and $\Omega_i$ are given by

$$a_3 = \Omega_1 = \Omega_2 = 0 \qquad \Omega_3 = \Omega \quad (11)$$

The determinantal equation then becomes

$$\begin{vmatrix} [\mu a^2 - \rho b^2 + (\lambda+\mu)a_1^2] & [(\lambda+\mu)a_1a_2 + 2\rho b\Omega] \\ [(\lambda+\mu)a_1a_2 - 2b\rho\Omega] & [\mu a^2 - \rho b^2 + (\lambda+\mu)a_2^2] \end{vmatrix} = 0$$

or

$$(b^2/a^2)^2 - (b^2/a^2)[(\mu/\rho) + \{(\lambda + 2\mu)/\rho\} - (4\Omega^2/a^2)] + (\mu/\rho)(\lambda + 2\mu)/\rho = 0 \quad (12)$$

Finally, solving for $b^2/a^2$ leads to the expression

$$b^2/a^2 = \tfrac{1}{2}\{[\mu/\rho] + [(\lambda + 2\mu)/\rho] - (4\Omega^2/a^2)\} \pm \{\tfrac{1}{4}([(\lambda + 2\mu)/\rho] - [\mu/\rho])^2 - (2\Omega^2/a^2)([(\lambda + 2\mu)/\rho] + [\mu/\rho]) + (4\Omega^4/a^4)\}^{1/2} \quad (13)$$

Equation (13) shows that the rotation introduces a coupling between the transverse and longitudinal waves The nature of this coupling is revealed more explicitly, however, by considering $\Omega^2/a^2$ small as compared with $\mu/\rho$ and by expanding the radical This leads to the equations

$$b^2/a^2 \approx [(\lambda + 2\mu)/\rho] - 4[(\lambda + 2\mu)/(\lambda + \mu)](\Omega^2/a^2) \quad (14)$$

and

$$b^2/a^2 \approx [\mu/\rho] + 4[(\lambda + 2\mu)/(\lambda + \mu)](\Omega^2/a^2) \quad (15)$$

These equations show that the propagation is still of a longitudinal and transverse nature, but for $a_i, b$ real (amplitude waves), the rotation tends to increase the velocity of the transverse-type waves while decreasing the velocity of the longitudinal type waves For $a_i, b$ imaginary (phase waves), the situation is reversed This latter case is an indication of the effect of rotation on the natural longitudinal and transverse vibration frequencies in rotating elastic bodies

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# Derivation of Element Stiffness Matrices


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RECENT derivation of element stiffness matrices of axisymmetrical shells by Grafton and Strome[1] is an example of the direct determination of stiffness matrices through the principle of virtual displacement[2,3] In applying this method, the displacement of a structural element is first expressed in terms of $n$ undetermined coefficients, where $n$ is the same as the number of the generalized displacement of the structural elements In general, for such analysis, the corresponding stress distribution of the element will not satisfy the equations of equilibrium The present note is to show that the displacement function may be assumed to contain more than $n$ undeterminate coefficients, and the employment of the principle of minimum potential energy enables the evaluation of these additional coefficients Solutions obtained by taking more terms in the displacement function will represent an improvement in the equilibrium conditions

One begins by considering a displacement vector $\{u\}(=\{u, v, w\})$ that contains $(n + l)$ undeterminate coefficients $\{\alpha\}$ as follows:

$$\{u\} = [A] \underset{(n+l)\times 1}{\{\alpha\}} \quad (1)$$


Received December 31, 1963 This work was supported by the U S Air Force under Contract No AF 04(694) 427.



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where the terms of matrix $[A]$ are functions of the coordinates $x$, $y$, and $z$ By using the strain displacement relations, one can write the strain distribution as

$$\{\epsilon\} = [W]\{\alpha\} \tag{2}$$

and when the stress-strain relation

$$\{\sigma\} = [E]\{\epsilon\} \tag{3}$$

is introduced, one can express the internal strain energy as

$$U = \tfrac{1}{2}\lfloor\alpha\rfloor[G]\{\alpha\} \tag{4}$$

where

$$[G] = \int_V [W^T][E][W]dV \tag{5}$$

From Eq (1), one can also express the $n$ generalized displacements $\{q\}$ at the node points in terms of the undetermined coefficients $\{\alpha\}$:

$$\underset{n\times 1}{\{q\}} = \underset{n\times(n+l)}{[B]}\ \underset{(n+l)\times 1}{\{\alpha\}} \tag{6}$$

By partitioning the matrix $[B]$ one can write

$$\underset{n\times 1}{\{q\}} = \underset{n\times n}{[B_a]}\underset{n\times 1}{\{\alpha_a\}} + \underset{n\times l}{[B_b]}\underset{l\times 1}{\{\alpha_b\}} \tag{7}$$

One can solve for $\{\alpha_a\}$ in terms of $[q]$ and $\{\alpha_b\}$ and write

$$\{\alpha\} = \begin{Bmatrix}\alpha_a \\ \alpha_b\end{Bmatrix} = \begin{bmatrix} \underset{(n\times n)}{B_a^{-1}} & \underset{(n\times l)}{-B_a^{-1}B_b} \\ \underset{(l\times n)}{0} & \underset{(l\times l)}{I}\end{bmatrix}\begin{Bmatrix} q \\ \alpha_b\end{Bmatrix} \tag{8}$$

Let

$$[M] = \begin{bmatrix} B_a^{-1} & -B_a^{-1}B_b \\ 0 & I \end{bmatrix} \tag{9}$$

The strain energy $U$ can thus be written as

$$U = \tfrac{1}{2}\lfloor q,\alpha_b\rfloor[K]\begin{Bmatrix} q \\ \alpha_b\end{Bmatrix} \tag{10}$$

where

$$[K] = [M^T][G][M] \tag{11}$$

The total potential energy including the work done by the generalized forces $\{Q\}$ is

$$\pi_p = U - \lfloor q\rfloor\{Q\} \tag{12}$$

The condition of minimum potential energy [i e , $\partial\pi_p/\partial q_i = 0$ ($i = 1, \ldots, n$), and $\partial\pi_p/\partial\alpha_{bj} = 0$ ($j = 1, \ldots, l$)] yields

$$\begin{bmatrix} \underset{(n\times n)}{K_{aa}} & \underset{(n\times l)}{K_{ab}} \\ \underset{(l\times n)}{K_{ba}} & \underset{(l\times l)}{K_{bb}}\end{bmatrix}\begin{Bmatrix} q \\ \alpha_b\end{Bmatrix} = \begin{Bmatrix} Q \\ 0\end{Bmatrix} \tag{13}$$

in which the $[K]$ matrix has been partitioned It is seen that $\{\alpha_b\}$ can be expressed in terms of $\{q\}$ by solving the last $l$ equations, and, after eliminating $\{\alpha_b\}$, the following equation results:

$$([K_{aa}] - [K_{ab}][K_{bb}^{-1}][K_{ba}])\{q\} = \{Q\} \tag{14}$$

By definition, the element stiffness matrix is

$$[k] = [K_{aa}] - [K_{ab}][K_{bb}^{-1}][K_{ba}] \tag{15}$$


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# Chemical Kinetic Analysis of Rocket Exhaust Temperature Measurements


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Recent investigations[1–3] of nozzle performance and flow conditions with finite-rate chemical reactions indicate that, if the reaction paths and rate constants are known, then the flow parameters can be predicted to a significantly higher degree of accuracy than by thermodynamic methods alone Conversely, it is sometimes possible to infer approximate values of the rate constants of the controlling reactions for a particular hypothesized mechanism from a comparison of theoretical predictions with experimental flow parameters Studies of this nature have been performed by Franciscus and Lezberg[4] and by Hoglund, Byron, and Carlson[5] There is a considerable difference between the values of rate constants which these two groups found best to fit their respective experimental data The differences in rate constants, under some conditions, lead to significant differences in predicted exhaust conditions

During the past few years, a technique of studying the infrared emissivities of gases, using a small rocket motor with a contoured nozzle to generate the hot gases, has been developed by Ferriso and his co-workers[6 7] In the course of these studies the exhaust temperatures of the rocket, operated with different propellants and various mixture ratios, were measured by an infrared emission-absorption technique[8] Temperatures between 530° and 2450°K have been measured by this method to an estimated accuracy of 50°K The rocket "burner" was of nominal 150-lb thrust, water-cooled, and of apparently high combustion efficiency (>95% theoretical $C^*$) The nozzle was designed by the Foelsch method[9] to produce a homogeneous, axially directed exhaust jet at 1 atm static pressure Photographs and shadowgraphs[10] of the exhaust show no shocks if the exit pressure and ambient pressure are balanced (by adjusting the propellant flow rate) Measurements are made less than 2 mm downstream of the exit plane The homogeneity of the gas sample at this region is shown by the constancy of measured total pressure across the exit plane and by the good agreement between gaseous spectral emissivities measured under these conditions and in absorption cells

Exhaust temperatures of this small motor, operated with RP-1 and gaseous $O_2$ at mixture ratios from 2 2 to 6, were measured under balanced conditions Measured temperatures fall between the values calculated for equilibrium and frozen flow The measured and thermodynamically calculated temperatures (solid lines) are shown in Fig 1

Figure 1 also shows values of the exhaust temperature calculated by applying a sudden-freezing model to the recombination reactions Analyses indicate that the energy release attendant upon the recombinations dominates the total energy release in the expanding flow and that other reactions are, to a good approximation, thermally insignificant The reaction scheme is the same used by Hoglund et al Three different sets of rate constants were employed: 1) those suggested by Hoglund et al from flame data; 2) those selected by Franciscus and Lezberg from a combination of shock-tube work and flame data; and 3) those representing the "lower limit" of published values


Received December 27, 1963 The author wishes to express his thanks to C C Ferriso and C B Ludwig for several helpful discussions of their data and for the use of those data here This work was supported by the Advanced Research Projects Agency through the U S Air Force Space Systems Division Contract No AF 04(695)-140



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